

Special Relativity: Some Notes

§ Length Contraction §

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➤ Notes to self in a little quest to understand some of the basics of special relativity.

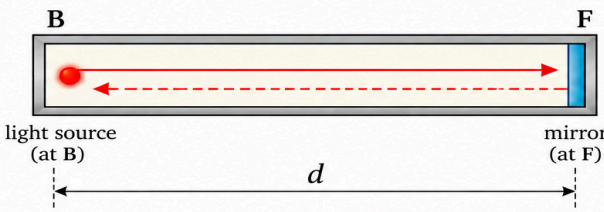
Somewhat analogous to the *time dilation* discussed in previous notes, but no less counterintuitive, is the principle of *length contraction*. Just as *time*, as measured by a stationary observer of a moving body, slows as compared with its measurement on the moving body itself, the actual *length* of the moving body is measured as being shorter as well, in the direction of motion.

Very odd indeed. But this is also the surprising outcome of Einstein's *special theory of relativity*. These notes aim to demonstrate this using basic (high school) maths. It's assumed the reader is already familiar with the aforementioned notes on time dilation (in particular with regard to some terminology introduced there, as well as the definition of the time dilation formula also derived there).

The Proof

For this exercise we'll consider a long horizontal box with a light source at the back (**B**) of the inside, and an inward facing mirror at the front (**F**). Imagine a light beam being emitted from the back toward the mirror at the front, bouncing off, and returning to the back. And we further imagine that this entire box is moving to the right at constant velocity **v**. We'll first consider how the length of this box is measured from the reference frame (**M**) of the moving box itself, and then from the reference frame (**S**) of an observer which is stationary relative to it.

In the box's rest frame (M)
(proper length d)



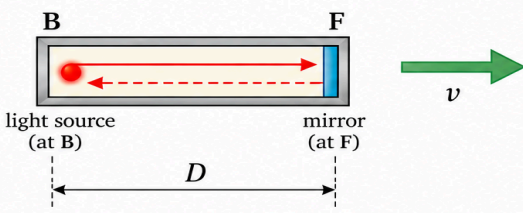
light source (at B) mirror (at F)

d

In the box's own rest frame (M), the light travels from B to F and back again. The round-trip time measured by the clock at B is

$$t = \frac{2d}{c}.$$

In the stationary observer's frame (S)
(box moving to the right, length contracted)



light source (at B) mirror (at F)

D

v

Observers in frame (S) measure a shorter length D . The light still makes the same round trip, and the total time measured by the clock at B is

$$T = \frac{2D}{c-v} + \frac{2D}{c+v}$$
$$= \frac{2Dc}{c^2 - v^2}.$$

If we refer to the length of the box as **d**, from the frame of reference (**M**) of the box itself, the light will have travelled from the back to the front to the back again in time **t**, which can be expressed simply

as (recalling that time is distance divided by rate; the rate being speed of light **c**; and the distance being twice the length of the box **d**):

$$t = \frac{2d}{c} \Rightarrow \text{take note of this for use below}$$

⊕ It may seem odd that, in trying to understand length contraction, we begin by talking about time – specifically, the time it takes light to travel from the back of the moving box to the front and back again. But the two are connected: the time required for the light to make these journeys depends on the distance between the back and front. Our ultimate goal is to relate distance as measured by two different frames of reference (one stationary and one moving relative to it). We already know, from our previous notes on time dilation, how the elapsed time measured in one reference frame relates to the elapsed time measured in another. We can therefore use what we already know about time to discover how the distance between the back and front must differ between the two reference frames.

From the reference frame (**S**) of the stationary observer of this moving box, let's call **T_F** the time it takes the light to go from the back (**B**) of the moving body to the front (**F**). Since the front (**F**) of the body is moving to the right, away from the light moving toward it, the stationary observer measures that the effective rate of the light moving toward **F** is the speed of light **c** minus the speed of the moving body **v**. Given this we have (recalling that time is distance divided by rate):

$$T_F = \frac{D}{c - v}$$

Similarly, from the reference frame (**S**) of the stationary observer, let's call **T_B** the time it takes the light to go from the front (**F**) of the moving body back to the back (**B**). Since the back (**B**) of the body is moving to the right, *toward* the light moving back toward it, the stationary observer measures that the effective rate of the light moving toward **B** is the speed of light **c** plus the speed of the moving body **v**. Given this we have (recalling that time is distance divided by rate):

$$T_B = \frac{D}{c + v}$$

⊕ A fastidious reader may be troubled by the above subtraction/addition of **v** and **c**, given it's been repeatedly stated that the speed of light (**c**) is constant from all points of view; i.e. from the point of view of **F**, for example, it is true that light is approaching it at speed **c** (even though **F** is moving away from the light). But we're talking about the stationary observer's point of view (reference frame **S**); they observe both the light moving toward **F** at speed **c**, and **F** moving away from the light at speed **v**, so it is appropriate to subtract off the **v** from the **c**. Similarly for the addition of **v** to **c** for the **B** case.

So the total time **T** it takes, from the stationary observer reference frame (**S**), for the light to go from the back of the moving body to the front and then back again is **T_F + T_B** thus:

$$T = \frac{D}{c - v} + \frac{D}{c + v} \Rightarrow \text{common denominators} \Rightarrow \frac{D(c + v)}{(c - v)(c + v)} + \frac{D(c - v)}{(c - v)(c + v)}$$

Now doing some algebraic transformations on this:

$$\begin{aligned} T &= \frac{D(c + v) + D(c - v)}{(c - v)(c + v)} && :: \text{add the fractions} \\ T &= \frac{Dc + Dv + Dc - Dv}{c^2 - v^2} && :: \text{simplify numerator and denominator} \end{aligned}$$

$$T = \frac{2Dc}{c^2 - v^2} \quad :: \text{ simplify numerator } \Rightarrow \text{ take note of this for use below}$$

But from results derived previously in our notes on time dilation (and taking care that we are using t and T to represent the same values in both sets of notes), we know that:

$$T = t\gamma \quad :: \text{ time dilation formula (see previous notes)}$$

Where γ (Greek letter gamma) is the Lorentz factor, also derived in our notes on time dilation – we'll continue referring to this formula as γ for brevity in what follows:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad :: \text{ restating the Lorentz factor}$$

$$\gamma^2 = \frac{1}{1 - v^2/c^2} \quad :: \text{ square of above } \Rightarrow \text{ take note of this for use below}$$

Substituting the definitions of T and t we took note of above, on the left and right, respectively :

$$\frac{2Dc}{c^2 - v^2} = \left(\frac{2d}{c} \right) \gamma \quad :: \text{ left side is } T \text{ and } 2d / c \text{ on right side is } t \text{ from above}$$

Now doing some more algebraic transformations on this:

$$\frac{Dc^2}{c^2 - v^2} = d\gamma \quad :: \text{ multiple by both sides by } c / 2$$

$$D \left(\frac{c^2}{c^2 - v^2} \right) = d\gamma \quad :: \text{ factor out } D \text{ from the left side}$$

$$D \left(\frac{c^2}{c^2(1 - v^2/c^2)} \right) = d\gamma \quad :: \text{ factor out } c^2 \text{ from the fraction on the left side}$$

$$D \left(\frac{1}{1 - v^2/c^2} \right) = d\gamma \quad :: \text{ cancel out } c^2 \text{ from the fraction on the left side}$$

$$D\gamma^2 = d\gamma \quad :: \text{ recognize the left side fraction as } \gamma^2 \text{ as noted above and swap in}$$

$$D = \frac{d}{\gamma} \quad :: \text{ solve for } D \text{ by dividing both sides by } \gamma^2 \text{ and reducing } \gamma / \gamma^2 \text{ to } \gamma$$

We end up with a relationship between the length of the moving body d from the reference frame ($\mathbf{M}$) of the moving body (i.e. its proper length), and the length D of that same body as it is measured by the reference frame ($\mathbf{S}$) of the stationary observer.

Notice that this nicely illustrates the inverse relationship between the time and length distortion as a factor of the Lorentz formula (γ). That is, as time is dilated or *increased* (i.e. multiplied) by the Lorentz factor, the length is (in the direction of motion) likewise reduced (i.e. divided) by the Lorentz factor; or written as such more explicitly:

$$T = t \times \gamma \quad :: \text{ time dilation formula; } T \text{ ranges from } t \text{ up to arbitrarily large multiple of } t$$

$$D = d \div \gamma \quad :: \text{ length contraction formula; } D \text{ ranges from } d \text{ down to arbitrarily small multiple of } d$$

Where, again, the Lorentz factor (γ) is defined as:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Alternative Explanation

Here's another way to understand this. Imagine a planet – let's call it *Novarus* – far from the Earth, and which can be considered stationary relative to the Earth (i.e. Earth and Novarus are at rest relative to each other, and therefore share the same reference frame). Let's use d to represent the distance between Earth and Novarus, as measured from their common frame of reference.

Now imagine a spacecraft travelling at (a high) constant velocity v from Earth toward Novarus. How long will it take for the spacecraft to reach Novarus? From Earth's frame of reference this is straightforward. Using T to represent this time (from Earth's reference frame), the spacecraft travels distance d at velocity v , so we have this (recalling that time equals distance divided by velocity):

$$\begin{aligned} T &= d / v &:: \text{time it takes spacecraft to reach Novarus according to Earth's reference frame} \\ d &= vT &:: \text{obviously from above dividing both sides by } v \Rightarrow \text{take note of this for use below} \end{aligned}$$

⊕ Note that we're using the name T here very purposefully. It relates to the way T was used in our time dilation notes, where it was used to represent the dilated time that a stationary observer measured on a body moving relative to it; and where we ended up with the formula $T = t\gamma$ which relates this T to the proper time t as measured in reference frame of the moving body itself.

And now consider the same journey from the *spacecraft's* frame of reference. In this case the spacecraft can regard itself as stationary, while the Earth is moving away from it, and Novarus is moving toward it, both at velocity v relative to the spacecraft. From this (the spacecraft's) reference frame, the time – let's call it t – that it takes to reach Novarus from Earth, can be expressed in terms of T above. This is because we've taken care to make T and t represent similar values as they did in our previous notes on *time dilation*, where we had this formula:

$$\begin{aligned} T &= t\gamma &:: \text{time dilation formula from previous notes where } \gamma \text{ is the Lorentz factor} \\ t &= T / \gamma &:: \text{obviously from above dividing both sides by } \gamma - \text{less than } T \text{ if } v \text{ greater than zero} \end{aligned}$$

⊕ Note that we're using the name t very purposefully. Recall that in our time dilation notes, t represented the proper time as measured in the reference frame of the moving body itself. Here, both events we're considering – the spacecraft's departure from Earth, and its arrival at Novarus – occur at the spacecraft itself from its point of view. The elapsed time between these events in the spacecraft's own reference frame is therefore proper time, so we use the same symbol t . Meanwhile, T is the corresponding elapsed time measured in the Earth's (and Novarus') reference frame, which views the spacecraft as moving. Thus these quantities have the same relationship established in our time dilation notes: $T = t\gamma$

And now we have an interesting problem. From the spacecraft's reference frame, Novarus is approaching the stationary spacecraft at the same relative speed v . Yet it reaches the spacecraft after only a lesser time t rather than the greater time T . The distance that Novarus has travelled toward the spacecraft in its own reference frame therefore cannot be d . Let us call this distance D

(from the spacecraft's reference frame). So we have this (recalling that distance equals velocity multiplied by time):

$$D = vt$$

And we can then substitute in our definition of t above:

$$D = v \left(\frac{T}{\gamma} \right)$$

Rearranging the above so we can easily recognize another equivalence:

$$D = \frac{vT}{\gamma}$$

We saw above that vT equals d so we can substitute this in, so we have:

$$D = d / \gamma$$

Just as we had using the first proof above.

⊕ The quantity names (not necessarily standardized in the literature) can get surprisingly confusing. We're using upper case letters to indicate the values "distorted" (loosely speaking) by the relativistic effect, relative to the corresponding proper values. That is, the dilated time is called T , and the contracted length is called D . And the lower case letters indicate the "normal" (loosely speaking) or proper values. That is, the proper time is t , and the proper length is d . The upper/lower case distinction most definitely does not indicate reference frame.

⊕ In other words: The value of T has been enlarged from t because the time (the measured interval between events) has increased; T is the measure of that larger time interval. Conversely, the value of D has been reduced from d because the distance (the measured space itself) has decreased; D is the measure of that reduced spatial distance.

⊕ Just remember: The proper time and proper length are one's own time and length as measured from one's own at-rest reference frame (or "rest frame"). And dilated time and contracted length are the corresponding time and length within another "moving" reference frame, as measured from one's own rest frame.

Relativistic Space Travel

From this alternative explanation we can readily consider a consequence related to potential long distance space travel. Assuming, for example, that our fictitious distant planet Novarus is 100 light-years away from Earth, we might naively think that, even travelling at near light speed, it would certainly take our astronauts on their spacecraft *at least* 100 years to get there.

But due to the results of length contraction (and time dilation), Novarus can theoretically be reached much more quickly. If we could (somehow) get the spacecraft to travel at about 98.06% the speed of light, then we could travel from Earth to Novarus in only about 20 years, from the point of view of the astronauts themselves on the spacecraft. Here's how we get this result from what we've learned.

Let 100 light-years be the distance d from Earth to Novarus. And let the spacecraft's proper time t be 20 years (the maximum duration our astronauts are willing to travel for). Then from the *spacecraft's*

own frame of reference, the distance **D** from Earth to Novarus is *contracted* as defined by our length contraction formula:

$$D = d / \gamma$$

But the spacecraft regards *itself* as stationary and *Novarus* as approaching it at velocity **v**, so we also have (recalling that distance equals velocity multiplied by time):

$$D = vt$$

Putting these two together we have:

$$d / \gamma = vt$$

Recalling that the Lorentz factor **γ** is a function dependent (only) on velocity **v**:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Let's swap this in explicitly:

$$d \sqrt{1 - v^2/c^2} = vt$$

Now do some algebraic transformations to solve for **v**:

$$\begin{aligned} d^2 (1 - v^2/c^2) &= t^2 v^2 && :: \text{square both sides} \\ d^2 - d^2 v^2 / c^2 &= t^2 v^2 && :: \text{distribute } d^2 \\ d^2 &= t^2 v^2 + d^2 v^2 / c^2 && :: \text{isolate } d^2 \\ d^2 &= v^2 (t^2 + d^2 / c^2) && :: \text{factor out } v^2 \text{ on the right side} \\ v^2 &= d^2 / (t^2 + d^2 / c^2) && :: \text{isolate } v^2 \\ v &= d / \sqrt{t^2 + d^2 / c^2} && :: \text{square root both sides} \end{aligned}$$

And finally with **v** solved for, we can plug in our actual numbers:

$$v = 100 / \sqrt{20^2 + 100^2 / 1^2} \quad :: 0.98058067569092$$

The **100** there is our 100 light-year distance from Earth to Novarus; the **20** is the time we want our astronauts to have to wait to get there; and why did we substitute **1** in for the speed of light **c**? Because in this example we're dealing with units of years for time and light-years for distance; light travels exactly one light-year in one year, therefore in these units **c** is **1** (light-year per year).

Calculating the **v** above, we get approximately **0.9806** light-years per year, which is the fraction of the speed of light **c** (i.e. 98.06% the speed of light **c**) our spacecraft has to travel at in order to reach

Novarus in 20 years, from the astronauts' point of view; a distance which from our Earth-bound perspective is 100 light-years.

⊕ *This is great for our astronauts, eager to get to Novarus. On the other hand however, back on Earth, their loved ones would sadly most likely be dead, since about 101.98 years will have passed there during their 20 year journey. This 101.98 years is simply derived from the speed v we've just calculated the spacecraft is travelling, and the distance d which the Earth measures from itself to Novarus, which is 100 light-years. And since time equals distance divided by velocity (as noted also previously $T = d / v$) we have $100 / 0.9806$ is about 101.98 years.*

⊕ *Or, we can get the same result using the time dilation formula $T = \gamma t$, where the Lorentz factor – which recall is $1 / \sqrt{1 - (v^2/c^2)}$ – is about 5.099 (given a velocity v of 0.9806 times c), and t is 19.61, so 5.099 times 19.61 giving 101.98.*

Thus we see that length contraction provides something *resembling* the practical effect of *faster-than-light* speed travel. The astronauts can reach a destination that is 100 light-years away from Earth in much less than 100 years of their *own* time (though 100+ years pass *on* Earth). But they never actually travel faster-than-light; in their own reference frame, it's the distance to the destination that has contracted. It remains true that literally travelling faster than light is impossible.

* * *

But wait a minute. If the astronauts traverse the 100 light-year distance in only 20 years, as just noted, this looks like faster-than-light speed. We know that from the astronauts point of view, they do not perceive faster-than-light travel because they are travelling a distance, from their perspective, of only about 19.6 light-years (from the Lorentz factor γ which is about 5.099 given the spacecraft speed v of 98.06% the speed of light, and given that according to the length contraction formula $D = d\gamma$, the distance of 100 light-years is reduced therefore by a factor of about 5.099).

From Earth's perspective however, do *they* view the spacecraft as travelling faster-than-light? No. From Earth's perspective, the spacecraft is travelling the full 100 light-years, and the time passing on Earth as we've seen is about 101.98 years, so they perceive the spacecraft is travelling at about 98% the speed of light. So not only does no one actually travel faster-than-light, no one *views* anyone as travelling faster-than-light either.

But I suppose this does mean that - in a *very qualified* way – we *can very loosely* say that faster-than-light speed travel *is* possible. Our astronauts were in fact, initially, stationary on Earth pondering a journey to a planet 100 light-years away. If they were naive to special relativity they might think – *surely there's no way to get there any sooner than 100 years, even travelling up to light speed it would take at least 100 years*. But as we've seen, with the length contraction, they can get there in substantially less time than that. Though it's important to keep in mind that they were *not* in fact actually travelling faster-than-light; their speed toward Novarus shortened the distance to it.

⊕ *By the way, why can't we actually travel faster than light c ? The maths give us a hint. Recall the Lorentz factor γ : $1 / \sqrt{1 - (v^2/c^2)}$... As the velocity v gets closer to c , the denominator approaches zero and γ grows without limit. When v itself reaches c , the expression breaks down entirely, with the denominator becoming zero, i.e. divide by zero, which is technically undefined (or more loosely, infinity). This has profound physical consequences involving energy and momentum. Accelerating any physical object with mass all the way to c would require an unlimited amount of energy (related to the famous $E = mc^2$). Thus a spacecraft can theoretically get arbitrarily close to light speed, but can never reach or cross it. Light itself presents a fascinatingly different case – one we'll leave for possible subsequent notes.*

Some Observations

One potential point of puzzlement with our first proof. Why do we have the light *returning* to **B**, rather than simply measuring its *one-way* trip from **B** to **F**? Well, a one-way measurement would require one clock at **B** to record the departure and another clock at **F** to record the arrival. We can't simply use the clock at **B** for both. The *information* that the light has arrived at **F** can't be transmitted instantaneously back to **B** – *information* itself can't travel faster than the speed of light (an important point in itself). Sending that information back to **B** therefore gives us effectively the same sort of round-trip we're using. And with two clocks (at **B** and **F**) we'd have to know that the two spatially separated clocks were properly synchronized – as we saw in our notes on non-simultaneity, synchronization between spatially separated clocks is reference-frame dependent. With the round-trip, we can use a *single* clock at **B**, thereby avoiding the synchronization problem entirely.

This, by the way, is closely related to the difficulty physicists have in actually measuring the *one-way* speed of light. If light leaves **B** and returns to **B**, a single clock can measure the complete round-trip time. But measuring only the trip from **B** to **F** requires assigning a departure time at one location and an arrival time at another – and therefore requires some way of synchronizing clocks at those two locations, which we know to be problematic due to non-simultaneity in special relativity. It is a surprisingly tricky and subtle problem; various clever techniques to synchronize two spatially separated clocks to do the measurements can be proposed, but they ultimately suffer deficiencies. There's an interesting YouTube video by *Veritasium*, with the (click-baity) title – *Why No One Has Measured The Speed Of Light* – which discusses this.

⊕ Fun fact: The speed of light has been measured to such extraordinary precision that, since 1983, it has actually been used to define the length of a meter, rather than the other way around. Previously (1960 to 1983), the meter was defined using a particular wavelength of light (and much earlier, 1889 to 1960, by a physical platinum-iridium meter bar kept in France). Today, the speed of light in a vacuum is defined to be exactly 299,792,458 meters per second (no decimals). Consequently, one meter is defined as the distance light travels in a vacuum in exactly $1 / 299,792,458$ th of a second.

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It's worth noting that in our previous notes on time dilation, in the light clock example, length contraction does *not* come into play at all. Length contraction occurs only *in the direction of motion*. Since our light clock was moving *horizontally* while the distance between its mirrors were vertical – *perpendicular* to that motion – the distance between the mirrors was the same for both observers. Thus our derivation of time dilation did not ignore length contraction as a simplifying assumption; there simply was no length contraction of the mirror separation to account for.

In other situations, however, time dilation and length contraction work together, each reinforcing rather than counteracting the strange consequences of the other. They are really two closely related aspects of the same relativistic effect, rather than two separate effects that simply multiply together.

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We can also now relate length contraction more directly to time dilation, and see how these are two complementary ways of describing the same journey, and how they fit together quantitatively.

Our relativistic space travel example was explained in terms of length contraction. The distance **D** (19.61 light-years) to Novarus, as observed by the astronauts, is shortened from the distance **d** (100 light-years), as observed by the Earth, by a factor of the *inverse* of the Lorentz factor **γ** (which is about 5.099 based on a spacecraft velocity **v** of about 98.06% the speed of light); i.e. by making use of our length contraction formula we derived above **D = d / γ**.

But notice how neatly this agrees with time dilation. From the astronauts' perspective, the journey covers about 19.61 light-years and takes 20 years. Dividing those values gives us the same velocity of 98.06% the speed of light. And the two elapsed times exactly satisfy our time dilation formula $T = \gamma t$: the astronauts' 20 years multiplied by γ (5.099) gives the Earth's 101.98 years.

So the two perspectives disagree about both the distance travelled and the time elapsed, but by corresponding amounts (time by the Lorentz factor, and distance by its inverse). The astronauts measure a shorter distance and a shorter elapsed time; the Earth measures a longer distance and a longer elapsed time. Yet each, using its own measurements of distance and time, calculates the same relative velocity.

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Another subtle but important point to understand is that length contraction does not apply just to *things*, material objects. It applies to the measurement of *space* itself, and therefore distances between *things* within it. We treated it implicitly as such in the alternate explanation above with our journey to Novarus, where we were measuring the (empty) *space* between the spacecraft and Novarus, and saw how its length (the *distance* between them) itself was contracted, from the spacecraft's frame of reference, thereby shortening the distance it has to travel to get there (when travelling at very high velocity).

Perhaps also not obvious is the fact that length contraction applies not only in the *forward* direction of motion but in the backward direction as well. More simply, the effect encompasses the expanse of space *parallel* to the direction of motion.

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As much as anything, the long distance space travel scenario is *truly* astonishing. Even given the *vast* expanse of our universe, we could *theoretically* travel to far distant locales in a reasonable, human scale amount of time. For example, it is in fact possible (again, *theoretically*) to travel to a star (say) *ten billion* light-years away from Earth in only (say) *one year* – if we could (in this hypothetical case) travel at (plugging in the numbers to the formulas we’ve discussed) 99.9999999999999997449% the speed of light. That’s a lot of nines – ridiculously close to the actual speed of light. Though anything like this is well out of reach, given our (current) technology, *nothing* in the laws of physics prevents it.