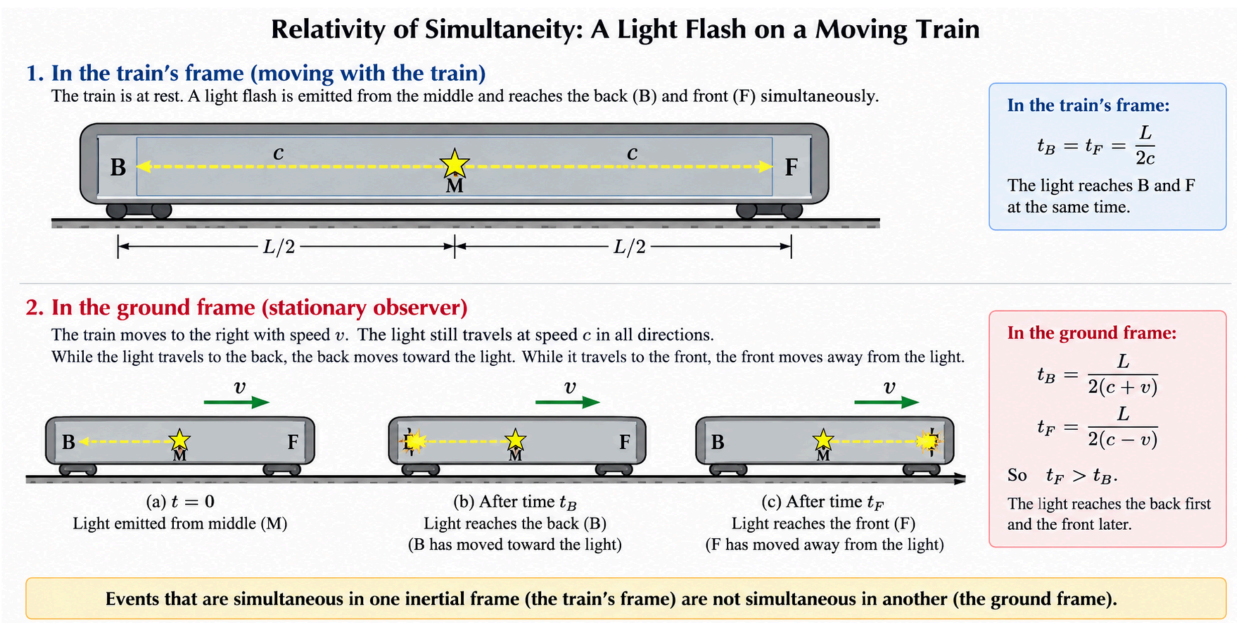
Special Relativity: Some Notes

§ Relativity of Simultaneity §

David Michaels – September 2026

➢ Notes to myself in a little quest to understand some of the basics of *special relativity*.

Consider a train car of length **L** moving to the right, relative to a stationary platform, at constant speed **v**. In the exact middle of this car is a bulb which flashes a light with rays emanating at the speed of light **c** towards the front **F** and the back **B** of the car. From the point of view (or *frame of reference*) of an observer *on* the train, the light reaches **F** and **B** *simultaneously*. But from the point of view of an observer on the platform, these events are *not* simultaneous; it will instead appear to the stationary observer that the light arrives *first* at **B** and *then* at **F**.



This apparent anomaly, the *relativity of simultaneity*, is a principle which says that observers moving at constant velocity relative to each other have differing ideas of simultaneous events. More generally, they also disagree about elapsed time: a clock moving relative to an observer is measured by that observer to run more slowly, an effect known as *time dilation*; the passage of time itself differs for these observers relative to each other. My “now” may differ from yours, as may our measurements of the passage of time, when we’re moving relative to each other.

This astonishing result was first articulated by Einstein in 1905 in his *special theory of relativity*. It is *special* because it applies primarily to bodies moving at *constant* velocity, without regard to acceleration, or gravity. He expanded this in 1915 in his *general theory of relativity* to work with acceleration (i.e. for observers moving in any way relative to each other), and with gravity; this ended up upending our conception of gravity itself – a whole other and more complicated story.

His insight was inspired largely (but not solely – a more complete explanation for Einstein's motivation is beyond the scope of these notes) by an experiment Michelson and Morley conducted in 1887 that helped provide the empirical basis for Einstein to postulate that the speed of light (**c**) is *constant* regardless of the motion of any observer. Michelson and Morley tried actually to detect the hypothetical *ether* through which the Earth was thought to move; this ether being *something* that was posited to permeate all of space providing a fixed medium through which light traveled; the experiment yielded a *null result*, i.e. they detected no effect of the Earth's expected motion through this ether. This (together with Maxwell's findings in 1865 regarding electromagnetism) points to a more complete explanation for Einstein's motivation; again, beyond the scope of these notes.

Einstein's assertion that the speed of light is constant regardless of motion was an extraordinarily counterintuitive proposition. Normally of course (when dealing with slower moving objects, like trains and balls and planets), motion is additive. For example, if a person on our moving train (moving at velocity **v**) throws a ball forward toward **F** at velocity **b**, then those on the platform would observe the ball moving at velocity **b + v**. This behavior fits intuitively with the physics established by Newton in 1687. But Einstein says that with light (**c**) rather than a ball (**b**), the light would move at velocity **c** (*not* **c + v**). This constancy of the speed of light did not mesh well with Newtonian physics; *something* was missing. Einstein put his finger on the precise nature of the problem, and a solution.

These notes simply demonstrate, using just basic maths (junior high school algebra), how our example with the moving train and beam of light flashing from its car, leads to the conclusion that there is no absolute, *frame-independent* simultaneity for spatially separated events; no universal time which applies to everyone, everywhere, regardless of how they are moving.

The Proof

Let us then, from the point of view of the observers on the platform, determine the time it takes for the light on the train car to reach each of **F** and **B** – we will refer to these times in the maths below as $\mathbf{t_F}$ and $\mathbf{t_B}$, respectively. As alluded to above, for observers on the train these will be equal, but for observers on the platform, these times will not – $\mathbf{t_F}$ will be *greater than* $\mathbf{t_B}$.

If $\mathbf{t_F}$ is the time it takes for the light to reach **F**, then $\mathbf{ct_F}$ is the distance the light must travel (from the middle of the car) to reach **F** in time $\mathbf{t_F}$ (recalling that distance is rate multiplied by time, the rate here being the speed of light **c**). What's this distance $\mathbf{ct_F}$? It's at least half the length of the train car (**L / 2**), *but* we also need to add in the distance travelled by the car (in the direction of **F**) during that time $\mathbf{t_F}$ which would be $\mathbf{vt_F}$, i.e. the speed of the train **v** times the time it takes for the light to reach **F** (recalling again that distance is rate multiplied by time). Thus we have:

$$ct_F = \left(\frac{L}{2}\right) + vt_F$$

to which we can apply these algebraic transformations:

$$\Rightarrow \; ct_F - vt_F = \left(\frac{L}{2}\right)$$

$$\Rightarrow \; t_F(c - v) = \left(\frac{L}{2}\right)$$

$$\Rightarrow \; t_F = \frac{L}{2(c-v)}$$

We can similarly find the time it takes for the light to reach **B**, which we call $\mathbf{t_B}$. This is analogous to $\mathbf{t_F}$, except that we *subtract* $\mathbf{vt_B}$ from **L / 2** rather than adding it (because **B** is moving toward rather than away from the approaching light). Thus we have:

$$ct_B = \left(\frac{L}{2}\right) - vt_B$$

to which we can apply these algebraic transformations:

$$\Rightarrow \; ct_B + vt_B = \left(\frac{L}{2}\right)$$
$$\Rightarrow \; t_B(c + v) = \left(\frac{L}{2}\right)$$
$$\Rightarrow \; t_B = \frac{L}{2(c+v)}$$

Now that we have the formulas for $\mathbf{t_F}$ and $\mathbf{t_B}$ we can compare them to see if (that) one ($\mathbf{t_F}$) is larger than the other. So we *subtract* $\mathbf{t_B}$ from $\mathbf{t_F}$ thus:

$$\Delta t = t_F - t_B$$
$$\Delta t = \left(\frac{L}{2(c-v)}\right) - \left(\frac{L}{2(c+v)}\right)$$

and apply these algebraic transformations:

$$\Rightarrow \; \Delta t = \frac{L}{2}\left(\frac{(c+v)}{2(c-v)(c+v)}\right)\left(\frac{(c-v)}{2(c-v)(c+v)}\right)$$
$$\Rightarrow \; \Delta t = \frac{L}{2}\left(\frac{(c+v)-(c-v)}{2(c-v)(c+v)}\right)$$
$$\Rightarrow \; \Delta t = \frac{L}{2}\left(\frac{2v}{c^2 + cv - cv - v^2}\right)$$
$$\Rightarrow \; \Delta t = \frac{L}{2}\left(\frac{2v}{c^2 - v^2}\right)$$
$$\Rightarrow \; \Delta t = \frac{vL}{c^2 - v^2}$$

This value $\mathbf{\Delta t}$ is the amount of *extra* (non-zero) time it takes for the light to reach **F**, as compared to the time it takes the light to reach **B**, from the point of view of the observers on the platform. And so we have a simple formula $\mathbf{\Delta t}$ to measure this difference in simultaneity for any body in our scenario of length **L** moving at velocity **v** relative to us.

<u>Some Implications</u>

Note that with **c** (the speed of light) being large (299,792,458 meters per second – over 670 million miles per hour), and $\mathbf{c^2}$ being *very* large indeed, and it being in the denominator of $\mathbf{\Delta t}$, *unless* **v** is also *very* large, the value of this formula $\mathbf{\Delta t}$ will be *very* small; meaning the effects of special relativity are virtually undetectable at all but *very* high speeds, as we well know from daily life where we can usually ignore these effects.

For example, if our train is 100 meters long (about 328 feet) and is travelling at 100 meters per second (about 224 miles per hour), then (plugging these values into the Δt formula) observers on the platform would see the light arriving at **F** just 11 quintillionths of a second *after* it arrived at **B** – so, yeah. Even increasing **v** to half the speed of light (over 335 million miles per hour), the difference (Δt) would only be 200 billionths of a second. In fact to get any humanly detectable difference we would need to increase **v** to 99.99999 percent the speed of light (that's five fractional nines), in which case the time difference (Δt) would be about 1.6 seconds.

We see then that as the velocity of our moving train **v** increases, the effects of the difference in simultaneity are more pronounced. But notice that we see this as well if we increase the *length* of the train **L**, since **L** in the numerator of Δt. So for example if we increase **L** to one million meters (about 62 miles - a *very* long train car), we would "only" need to increase **v** to 99.9 percent the speed of light (only one fractional nine) for us to get to the humanly detectable difference (Δt) of 1.6 seconds.

Also notice that if **v** is zero, i.e. if the train is not moving at all (or conversely, if the platform is moving along with the train at its speed, resulting in **v** being zero), then both the train and the observers on the platform would share the same frame of reference (they would be at rest relative to each other), in which case they would naturally share the same notion of simultaneity; which is reflected in Δt in which, the **v** being a multiplier in the numerator, the result is zero.

And then note that if our train is moving *at* the speed of light itself (a physical and theoretical impossibility), i.e. if we set **v** to the value of **c**, then the denominator of Δt evaluates to zero, which produces an undefined value (or "infinity", to be a little sloppy with the conceptual maths); meaning that the light will *never* reach **F**.

Simplification and Complication

These conclusions can actually also more simply be observed *without* going to the bother of subtracting t_B from t_F, just by examining the original formulas for these directly, and noting that obviously for non-zero values for **v**, these are distinctly different.

$$t_F = \frac{L}{2(c-v)}$$
$$t_B = \frac{L}{2(c+v)}$$

Note that if the speed of the train **v** is zero (or very small) then the time it takes the light to arrive at **F** and **B** are equal (or very close) to simply half the length **L** of the train divided by the speed of light **c** (recalling that time equals distance divided by rate), which makes intuitive sense.

On the other hand, if **v** is itself the speed of light (again, a physical and theoretical impossibility) then t_F evaluates to undefined or "infinity" (again with the sloppy conceptual maths), i.e the light will never reach **F**. And doing the same substitution for t_B, this evaluates to:

$$t_B = \frac{L}{4c}$$

It turns out that this is not a meaningful result given what has been discussed in these notes thus far. As noted, since a physical object (like a train) travelling at light speed is a physical and theoretical impossibility (a result of *special relativity* beyond the scope of these notes), then positing any such thing is not wholly sensible, given the scope of these particular notes.

The complication this raises, which these notes have not taken into account, is this: Just as observers moving relative to each other don't agree on whether or not spatially separated events are simultaneous, they *also* don't agree on their measurements of *length* itself. A moving object is measured to be *shorter* in the direction of its motion than it is when measured at rest – an effect known as *length contraction*. Thus even our seemingly straightforward statement that the train has length **L** requires us to specify who is measuring **L**: the observers on the train or those on the platform (so in the very first sentence of these notes then, it would be more rigorous to say that the length of our train car is **L** *as measured from the platform*).

This becomes increasingly important as **v** approaches **c**, since length contraction increases with velocity. Thus our casual substitution of **v = c**, while continuing to treat **L** as an ordinary fixed length, should not be expected to describe a physically meaningful situation. There is an intimate relationship among this length contraction, and time dilation, and the relativity of simultaneity; another consequence of special relativity, and may be considered separately.

What Would Newton Say?

It may be instructive to consider what Newton might have done with our example. According to Newtonian physics the light would act like any other (larger, slower) object. So starting from our original formula in the proof above for $\mathbf{t_F}$:

$$ct_F = \left(\frac{L}{2}\right) + vt_F$$

rather than **c** Newton would use **c + v** since (like a ball thrown on the train) the velocity of light would be additive to the velocity of the train. So we would instead have this:

$$(c + v)t_F = \left(\frac{L}{2}\right) + vt_F$$

and simplifying we see that the $\mathbf{vt_F}$ terms, being on both sides, cancel out:

$$\Rightarrow ct_F + vt_F = \left(\frac{L}{2}\right) + vt_F$$
$$\Rightarrow ct_F = \left(\frac{L}{2}\right)$$
$$\Rightarrow t_F = \left(\frac{L}{2c}\right)$$

That is, the time $\mathbf{t_F}$ it takes the light to reach **F** is simply one half the length **L** of the train car divided by the velocity of light **c** (recalling that time equals distance divided by rate).

For $\mathbf{t_B}$ it is exactly analogous to $\mathbf{t_F}$. We start with this from our previous proof:

$$ct_B = \left(\frac{L}{2}\right) - vt_B$$

we substitute **c** – **v** for **c** (minus here rather the plus above for the **F** side because the train is moving in the opposite direction of the light as it travels toward the **B** side):

$$(c - v)t_B = \left(\frac{L}{2}\right) - vt_B$$

and simplifying we see that the –$\mathbf{vt_B}$ terms, being on both sides, cancel out:

$$\Rightarrow ct_B - vt_B = \left(\frac{L}{2}\right) - vt_F$$
$$\Rightarrow ct_B = \left(\frac{L}{2}\right)$$
$$\Rightarrow t_B = \left(\frac{L}{2c}\right)$$

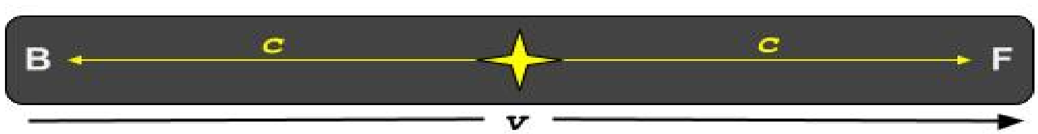
Again as with $\mathbf{t_F}$, the time $\mathbf{t_B}$ it takes the light to reach **B** is simply one half the length **L** of the train car divided by the velocity of light **c**. This is a very simple illustration of the broader fact that Einsteinian special relativity "degenerates" into normal Newtonian physics for "normal", slower moving bodies; which must have been a great comfort to Einstein.

<u>Another Observer</u>

So we have observers on the platform seeing the light on the train hitting **F** *after* **B**; whereas the observers on the train see these to be simultaneous. But we can add yet *another* observer that would see the light arriving at **F** *before* **B**.

Consider a motorcycle on the platform, moving parallel to and in the same direction as the train at velocity **2v** (twice the train speed), relative to the platform. For the observer on the motorcycle, the train would appear to be moving toward its rear, at velocity **v**.

The situation for the motorcycle is *exactly* the same as the platform, *except* the train appears to be moving to the *left* rather than the *right*. All of the analysis is exactly the same *except* we'd swap **F** and **B**. And the motorcycle concludes the light hits **F** first and then **B.**


B
c
c
F
v