Special Relativity: Some Notes

§ Time Dilation §

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> ➢ Notes to myself in a little quest to understand some of the basics of *special relativity*.

Imagine an odd sort of clock – a very tall transparent box of height **L** which has a down-facing mirror on its ceiling and an up-facing mirror on its floor. A light is emitted from the bottom toward the top, and the light simply bounces up and down between the mirrors, each up journey a *tick* and each journey down a *tock* – *tick-tock*. The duration of each tick-tock is double the height of the box divided by the speed of light (recalling that time is distance divided by rate).

For example, if the height of our box **L** is half the distance light travels in one second, i.e. 149,896,229 meters (the speed of light being 299,792,458 meters per second), then the time it takes for the light to travel from the bottom to the top (a *tick*) and then back down again (a *tock*), one tick-tock cycle, would be one second – i.e double the box height divided by the speed of light (recalling that time is distance divided by time).

Now what if we have two of these clocks, one being stationary where we are, and the other one that's moving horizontally (to the right, say) at (a high) velocity **v** relative to us. Would we measure the moving clock to be running (*tick-tocking*) at the same rate as our stationary one? The surprising answer turns out to be *No*. In fact, the stationary observer would measure the moving clock's time as running *slower* than its own stationary clock.

This is the astonishing outcome of the *special theory of relativity* that Einstein first articulated in 1905. This effect, called *time dilation*, says that clocks (and *everything*) run slower (i.e. time itself passes more slowly), when moving relative to, and when measured from the point of view of, a stationary observer. These notes aim to demonstrate this using basic (high school) maths.

Some Terminology

- A *reference frame* is essentially a point of view from which positions, distances, and times are measured. For example, a person standing on a stationary train platform and a person riding on a moving train (relative to that platform) are using two different reference frames.

- An *inertial reference frame* is a reference frame that is not accelerating – it's either at rest or moving in a straight line at constant velocity (note that non-straight movement is considered *acceleration*). Special relativity deals primarily with inertial reference frames. Importantly, there is no unique, absolute "stationary" inertial frame in and of itself; if we say "stationary observer" we mean stationary relative to another object moving at constant velocity (and we could equally say that the *moving* object is "stationary" and that the *stationary* one is "moving" – it's all relative with initial reference frames). For example, an observer on a train can view the train as stationary and the platform as moving backward, *or* the observer on a platform can view the platform as stationary with the train moving forward.

These notes concern only inertial reference frames, and any references here to "moving" or "stationary" observers are always assumed implicitly to refer to inertial reference frames.

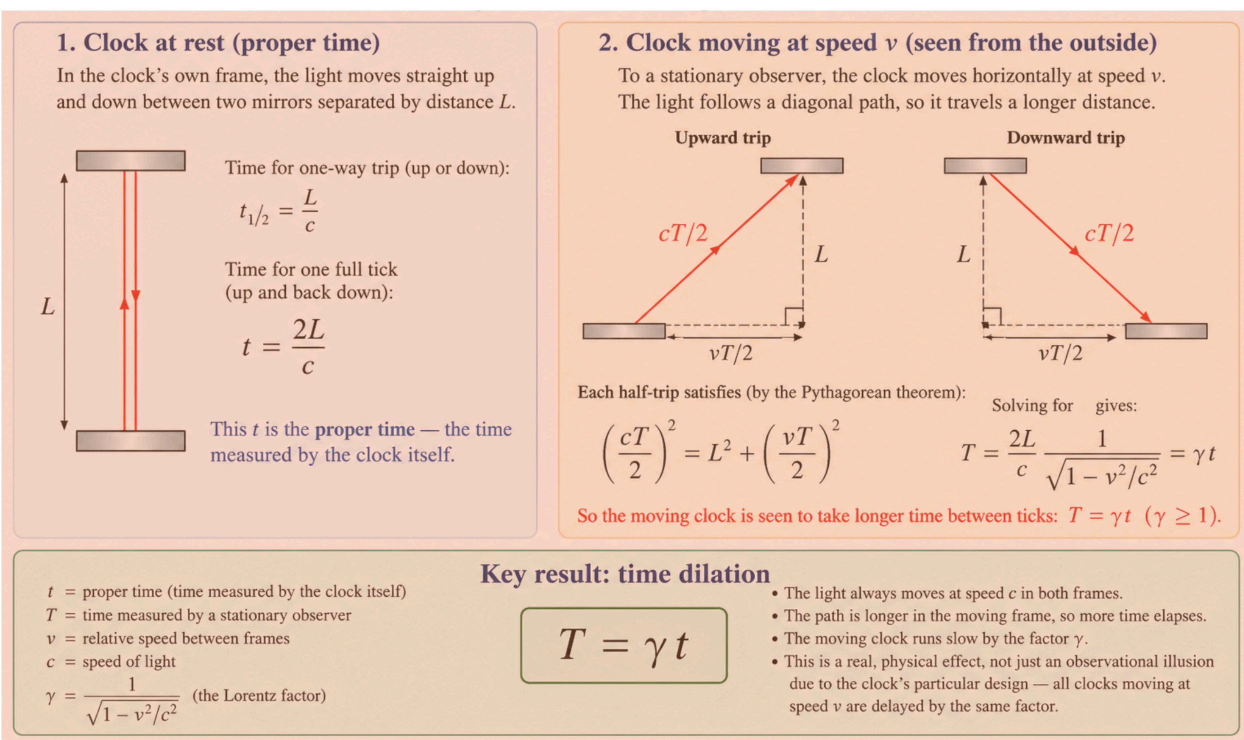
- The time measured in one's own inertial reference frame is called *proper time*. It's the time that passes for the clock from the clock's own point of view. If a light clock is moving relative to us, the time measured by an observer traveling *along with* that clock is the clock's proper time; same for a stationary observer; this proper time passes "normally".

- A stationary observer (relative to the moving clock), on the other hand, measures the moving clock from its own reference frame, and finds that it runs more slowly relative to clocks in its own reference frame; the passage of time as measured by clocks in the stationary observer's reference frame is known as *coordinate time*.

The Proof

First, for notational simplicity, we'll just consider the *tick* portion of the *tick-tock* cycle, i.e. what happens as the light travels from the bottom to the top of the box. The *tock* part of the cycle works *exactly* analogously to it – whatever is said about the *tick* works equally for the *tock*.

So we have two of these identical light clocks. One is stationary and the other is moving at constant velocity **v** relative to it. For brevity, in what follows, we'll refer to the reference frame of the stationary clock as **S**, and to the reference frame of the clock moving (relative to **S**) as **M**.



For an observer in (the moving) reference frame **M** the clock ticks (i.e. the light travels from the bottom to the top of the box) in what we'll call time **t**. This is called the *proper time* for **M**. From this reference frame **M** the clock runs normally. And this **t** can be expressed simply as the following (recalling that time is distance divided by rate; the distance being the height of the box **L**; and the rate being the speed of light **c**):

$$t = \frac{L}{c}$$

Think now about how the (stationary) observer in reference frame **S** sees the (moving) clock in reference frame **M**, as the light bounces up and down within the box. This observer in **S** will see the light moving in an upward *diagonal* line till it hits the top (the *tick*), and then in a downward diagonal line till it hits the bottom (the *tock* – but as mentioned let's ignore this *tock* and just consider the *tick*). We'll call the length of this upward diagonal **D**. The observer in reference frame **S** will measure the clock in reference frame **M** as ticking (i.e. as the light travels from the bottom to the top of the box) in what we'll call time **T**. And this **T** can be expressed simply as the following (recalling that time is distance divided by rate; and the rate being the speed of light **c**):

$$T = \frac{D}{c}$$

From this we see that **D** can also be expressed as (recalling that distance is rate times time):

$$D = cT$$

But **D** can *also* be described as the length of the hypotenuse of a right triangle, where the vertical side of the triangle is the height of our box **L**, and the horizontal side is the distance the box travels in time **T**, which would be the speed of the box **v** times **T**, that is **vT** (recalling that distance is rate times time).

Remembering the *Pythagorean Theorem* (from high school?) which says that the length of the hypotenuse of a right triangle equals the sum of the squares of the other two sides (ie. $\mathbf{x^2 + y^2 = z^2}$), we thus also have this equivalence for **D**:

$$D^2 = v^2T^2 + L^2$$

Putting these two definitions of **D** together we have:

$$c^2T^2 = v^2T^2 + L^2$$

Now let's do some algebraic transformations on this:

$$c^2T^2 - v^2T^2 = L^2$$

$$T^2\ (c^2 - v^2) = L^2$$

$$T = \frac{L}{\sqrt{c^2 - v^2}}$$

$$T = \frac{L}{\sqrt{c^2(1 - v^2/c^2)}}$$

Wait, in that last step, why are we factoring out a ***c²*** *in the denominator's square root?* Yes, that seems spurious. But it's because we know where we're going – roll with it – as we're going through this we're also going to derive something called the *Lorentz factor* (stay tuned). Continuing on:

$$T = \frac{L}{\sqrt{c^2}\,\sqrt{1 - v^2/c^2}}$$

$$T = \frac{L}{c\sqrt{1 - v^2/c^2}}$$

$$T = \frac{L}{c}\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right)$$

Now we see **L** divided by **c**. We've seen this before – above, where we said that this is the *proper time* **t** as observed from reference frame **M**. Substitute that in and we now get a relationship between the proper time **t** as observed in reference frame **M**, and the time **T** observed in reference frame **S**:

$$T = t\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right)$$

In other words, **t** is the duration of the tick as experienced by observers moving along with the moving clock (in reference frame **M**); and **T** is the duration of the tick of that same moving clock as measured by stationary observers (in reference frame **S**). And we see that these values for **t** and **T** are not equal (when **v** is non-zero, as we have in our case where the clock is moving).

And this is where we can define the *Lorentz factor*. That's the part in the parenthesis above; it's traditionally represented using the Greek letter gamma (γ) thus:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

So we end up with this very simple looking relationship between the two times **t** and **T**:

$$T = t\gamma$$

For math or computer types, this Lorentz factor γ *is actually a* function *with a single parameter which is velocity, so we might want to write it as* γ(*v*) *– but it is traditionally written as just* γ *with the functional part implied.*

This says that the time **T** is what we measure as passing for the moving clock (in reference frame **M**) from our stationary position (in reference frame **S**), and that it is related to the proper time **t** as measured from the moving clock's own point of view (from within its own reference frame **M**), by a factor of this Lorentz factor γ.

Stated even more plainly (if less precisely): This tells us how much time (**T**) has passed for a stationary observer compared to how much time (**t**) has passed for a body moving relative to it.

For simplicity we've had the clock moving (left to right) perpendicular to the stationary observer. What if it were moving at an angle, or straight out away from the stationary observer? It would work *exactly* the same. We'd just need to *rotate* the coordinates for our stationary frame of reference so that one axis was aligned with the direction of the clock's motion. The clock would then again appear to be moving horizontally across our coordinate system, and the same right-triangle geometry and derivation would apply. Rotating our coordinate system does *not* change our reference frame; we're merely choosing differently oriented axes with which to measure the same events.

<u>Some Observations</u>

One common point of confusion to clarify first. One might think we're saying that time is slowing down for the moving body, that the observer on the moving body itself would experience time in *slow motion*. No. For the observer moving *on* or *with* the moving body, time (their *proper time*) appears perfectly normal (as they're "at rest" relative to themselves – just like the stationary observer). It's from the point of view of the stationary observer (relative to the moving body), who is measuring the passage of time (using their own local stationary clock) on the moving body – that measurement is what is moving slower, as compared to the measurement of time using our local stationary clock.

That said, it really is true that for this stationary observer, when they view the moving body, they observe everything on that moving body as *really* moving slower, from their stationary point of view – it really would *look* like things were moving in *slow motion*. And not just clocks, but *everything* including (say) balls flying through the air, sound waves, and even biological processes; if in fact they could see into the clock, vehicle, or whatever body it is that's moving.

Back to our formula $T = t\gamma$ which describes the relationship between how time **T** passes from the stationary point of view (within reference frame **S**) when observing a body moving relative to it (in reference frame **M)**, within which time is passing in *proper time* **t**. The full non-Lorentz formula is:

$$T = t\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right)$$

Just as a sanity check, notice what happens if **v** is zero, i.e. if the moving clock is actually not moving at all. Then the $\mathbf{v^2/c^2}$ is zero, giving one in the denominator, and we'll ultimately be left with just **T = t**, i.e. no time dilation at all. Which obviously is true if the clock is not moving; it acts just like our stationary clock, since they are both in the same (stationary) reference frame.

Also notice that if **v** is actually the speed of light **c** (a physical and theoretical impossibility), then the denominator is zero, which yield an undefined result (or to be sloppy with the maths you might think of it as infinity - meaning, supposedly, infinite time dilation).

But if **v** is non-zero (and less than **c**), then notice that with the speed of light **c** being large (299,792,458 meters per second), and $\mathbf{c^2}$ being *very* large indeed, unless **v** is pretty big, the resulting time dilation will be negligible, at least in normal human terms.

Some Examples

For example, if **v** (the speed of the moving clock relative to a stationary observer) is 1000 meters per second (about 2236 miles per hour), and we observe the moving body for 60 seconds, then (plugging the numbers into our formula) we find that **T** is 60.0000000003338 seconds, meaning that 60 seconds (**t**) has passed for the moving body, but for the stationary observer 60 seconds *plus* 334 trillionths of a second have passed. Yeah, big deal.

To see an actual human scale difference we could increase **v**, for example, to 25% the speed of light (over 167 million miles per hour), in which case we would see (plugging in the numbers into our formula) that 60 seconds (**t**) for our moving travellers will be 62 seconds (**T**) for us stationary observers; if they travelled for a whole year, an extra 12 days would have passed.

The Twins Paradox

One of the most salient outcomes of time dilation is the *twin paradox*, where a person on Earth travels in a space craft (at high speed) away from (and relative to the Earth) and then returns again. The returning traveller finds that his twin on Earth is now an elderly person whereas the returnee is still young. This is in fact entirely possible as a direct result of the time dilation of special relativity discussed here. Time has passed for the traveler's clock (and *everything* aboard his space craft) at a slower rate than the Earth's clocks (and *everything* on the Earth).

This has become a pop culture trope, as seen perhaps most iconically, for example (though no actual twins involved), at the end of the movie *Planet of the Apes*, where Charlton Heston realizes that he's been on Earth the whole time. Due to his journey through space and back, he's aged only a bit, but Earth itself has aged hundreds of years; he's effectively time travelled into Earth's future. Although the observant moviegoer will recall that the Earth has aged 700 years and Heston and team have aged (i.e have been travelling for) 6 months, meaning they would have had to have been travelling at 99.9999745% the speed of light - incredible given any currently conceivable technology.

A more realistic, if less impressive example is the US astronaut Mark Kelly, who's now about 8.6 milliseconds younger than his twin brother Scott, due to Mark's extended stay on the International Space Station.

Objection to the Twins Paradox

One objection to this might be that, since the relativity of inertial reference frames means that astronauts on a spacecraft could validly regard themselves as at rest, with the Earth in motion relative to them, why wouldn't the situation be reversed, with the astronauts aging *more* than those on Earth? And as long as the astronauts and the Earth are simply moving apart at constant velocity, the situation is indeed symmetrical: each can regard the other as moving, and each measures the other's clock as running slowly.

But this symmetry is broken when the astronauts *turn around* and return to Earth. It's the astronauts who change their velocity, and therefore *change* from one inertial reference frame to another; the Earth remains in essentially the *same* inertial reference frame throughout the journey. When the astronauts finally return, the two clocks can be brought together and compared directly, and less time will have elapsed for the astronauts than for the people who remained on Earth.

Length Contraction

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Long Distance Space Travel

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